S14 C3(R)

1. Express

$$\frac{3}{2x+3} - \frac{1}{2x-3} + \frac{6}{4x^2-9}$$

as a single fraction in its simplest form.

$= (2x-3)(2x+3)$

(4)

$$\frac{3(2x-3) - (2x+3) + 6}{(2x+3)(2x-3)} = \frac{4x-6}{(2x+3)(2x-3)} = \frac{2\cancel{(x-3)}}{(2x+3)\cancel{(2x-3)}}$$

$$= \frac{2}{2x+3} \quad \#$$

2. A curve C has equation $y = e^{4x} + x^4 + 8x + 5$

(a) Show that the x coordinate of any turning point of C satisfies the equation

$$x^3 = -2 - e^{4x}$$

(3)

(b) On the axes given on page 5, sketch, on a single diagram, the curves with equations

(i) $y = x^3$,

(ii) $y = -2 - e^{4x}$

On your diagram give the coordinates of the points where each curve crosses the y-axis and state the equation of any asymptotes.

(4)

(c) Explain how your diagram illustrates that the equation $x^3 = -2 - e^{4x}$ has only one root.

(1)

The iteration formula

$$x_{n+1} = (-2 - e^{4x_n})^{\frac{1}{3}}, \qquad x_0 = -1$$

can be used to find an approximate value for this root.

(d) Calculate the values of x_1 and x_2, giving your answers to 5 decimal places.

(2)

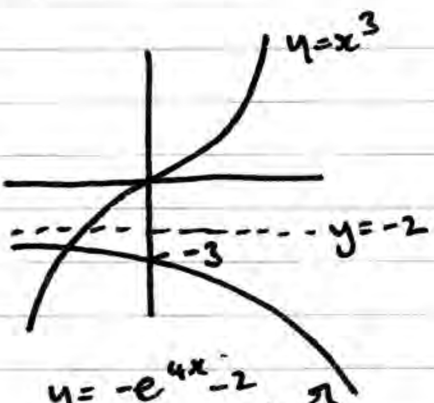
(e) Hence deduce the coordinates, to 2 decimal places, of the turning point of the curve C.

(2)

a) $\frac{dy}{dx} = 4e^{4x} + 4x^3 + 8$ TP $\frac{dy}{dx} = 0 \Rightarrow 4x^3 = -8 - 4e^{4x}$

$\therefore x^3 = -2 - e^{4x}$

b)



c) Only one point of intersection.

d) $x_0 = -1$

$x_1 = -1.26376$

$x_2 = -1.26126$

e $(-1.26, -2.55)$

3. (i) (a) Show that $2 \tan x - \cot x = 5 \operatorname{cosec} x$ may be written in the form

$$a \cos^2 x + b \cos x + c = 0$$

stating the values of the constants a, b and c.

(4)

(b) Hence solve, for $0 \leqslant x < 2\pi$, the equation

$$2 \tan x - \cot x = 5 \operatorname{cosec} x$$

giving your answers to 3 significant figures.

(4)

(ii) Show that

$$\tan \theta + \cot \theta \equiv \lambda \operatorname{cosec} 2\theta, \qquad \theta \neq \frac{n\pi}{2}, \quad n \in \mathbb{Z}$$

stating the value of the constant λ.

(4)

a) $\frac{2\sin x}{\cos x}{}^{\times \sin x}_{\times \sin x} - \frac{\cos x}{\sin x}{}^{\times \cos x}_{\times \cos x} = \frac{5}{\sin x}{}^{\times \cos x}_{\times \cos x} = \frac{2\sin^2 x - \cos^2 x}{\sin x \cos x} = \frac{5\cos x}{\sin x \cos x}$

$\therefore 2(1-\cos^2 x) - \cos^2 x = 5\cos x \Rightarrow 3\cos^2 x + 5\cos x - 2 = 0$

$a = 3 \quad b = 5 \quad c = -2$

b) $(3\cos x - 1)(\cos x + 2) = 0 \Rightarrow \cos x = \frac{1}{3} \quad x = \cos^{-1}\left(\frac{1}{3}\right)$

No solution

$x = 1.23^c, 5.05^c$

ii) $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta}$

$= \frac{2}{2\sin\theta\cos\theta} = \frac{2}{\sin 2\theta} = 2\operatorname{cosec}^2 2\theta \qquad \lambda = 2$

4. (i) Given that

$$x = \sec^2 2y, \qquad 0 < y < \frac{\pi}{4}$$

show that

$$\frac{dy}{dx} = \frac{1}{4x\sqrt{(x-1)}}$$

(4)

(ii) Given that

$$y = (x^2 + x^3)\ln 2x$$

find the exact value of $\frac{dy}{dx}$ at $x = \frac{e}{2}$, giving your answer in its simplest form.

(5)

(iii) Given that

$$f(x) = \frac{3\cos x}{(x+1)^{\frac{1}{3}}}, \qquad x \neq -1$$

show that

$$f'(x) = \frac{g(x)}{(x+1)^{\frac{4}{3}}}, \qquad x \neq -1$$

where g(x) is an expression to be found.

(3)

i) $x = (\sec 2y)^2 \qquad \frac{dx}{dy} = 2(\sec 2y) \times 2\sec 2y \tan 2y$

$= 4\sec^2 2y \tan 2y$

$\frac{\sin^2 y}{\cos^2 y} + \frac{\cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y} \qquad \therefore \frac{dy}{dx} = \frac{1}{4\sec^2 2y \tan 2y}$

$\tan^2 y + 1 = \sec^2 y$

$\tan y = \sqrt{\sec^2 y - 1} \qquad \therefore \frac{dy}{dx} = \frac{1}{4x\sqrt{x-1}}$

ii) $u = x^2 + x^3 \qquad v = \ln 2x \qquad \frac{dy}{dx} = (2x+3x^2)\ln 2x + \frac{x^2+x^3}{x}$

$u' = 2x + 3x^2 \qquad v' = \frac{2}{2x} = \frac{1}{x} \qquad = x(2+3x)\ln 2x + x + x^2$

$x = \frac{e}{2} \qquad \ln 2x = \ln e = 1 \quad \Rightarrow \frac{dy}{dx}\Big|_{\frac{e}{2}} = \frac{e}{2}\left(2 + \frac{3e}{2}\right) + \frac{e}{2} + \frac{e^2}{4}$

$= \frac{3}{2}e + \frac{4}{4}e^2 = \frac{3}{2}e\left(1 + \frac{2}{3}e\right)$

(iii) $u = 3\cos x \quad v = (x+1)^{\frac{1}{3}}$

$u' = -3\sin x \quad v' = \frac{1}{3}(x+1)^{-\frac{2}{3}}$

$$\frac{-3(x+1)^{\frac{1}{3}}\sin x - (x+1)^{-\frac{2}{3}}\cos x}{(x+1)^{\frac{2}{3}}}$$

$$= \frac{(x+1)^{-\frac{2}{3}}\left[-3(x+1)\sin x - \cos x\right]}{(x+1)^{\frac{2}{3}}}$$

$$= \frac{-3(x+1)\sin x - \cos x}{(x+1)^{\frac{4}{3}}}$$

5. (a) Sketch the graph with equation

$$y = |4x - 3|$$

stating the coordinates of any points where the graph cuts or meets the axes.

(2)

Find the complete set of values of x for which

(b)

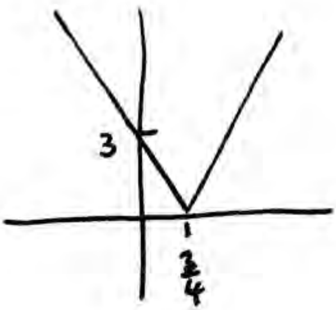
$$|4x - 3| > 2 - 2x$$

(4)

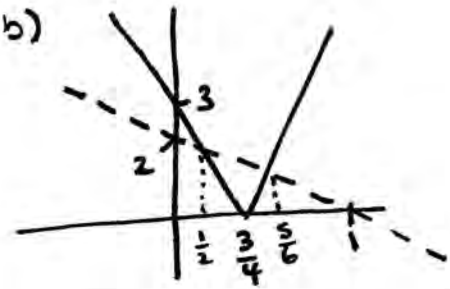
(c)

$$|4x - 3| > \frac{3}{2} - 2x$$

(2)



b)



$4x - 3 = 2 - 2x$
$6x = 5$
$x = \frac{5}{6}$

$4x - 3 = 2x - 2$
$2x = 1$
$x = \frac{1}{2}$

$x < \frac{1}{2}$ or $x > \frac{5}{6}$

c) $4x - 3 = \frac{3}{2} - 2x$
$6x = \frac{9}{2}$
$x = \frac{9}{12} = \frac{3}{4}$

$x \in \mathbb{R}$
$x \neq \frac{3}{4}$

6. The function f is defined by

$$f : x \rightarrow e^{2x} + k^2, \quad x \in \mathbb{R}, \quad k \text{ is a positive constant.}$$

(a) State the range of f. **(1)**

(b) Find f^{-1} and state its domain. **(3)**

The function g is defined by

$$g : x \rightarrow \ln(2x), \quad x > 0$$

(c) Solve the equation

$$g(x) + g(x^2) + g(x^3) = 6$$

giving your answer in its simplest form. **(4)**

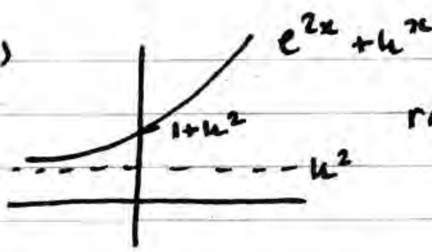
(d) Find fg(x), giving your answer in its simplest form. **(2)**

(e) Find, in terms of the constant k, the solution of the equation

$$fg(x) = 2k^2$$

(2)

a)



range $y > k^2$

b) $x = e^{2y} + k^2 \Rightarrow e^{2y} = x - k^2 \Rightarrow 2y = \ln|x - k^2|$

$\therefore y = \frac{1}{2}\ln|x - k^2| = f^{-1}(x)$

$x > k^2$
domain

c) $g(x) + g(x^2) + g(x^3) = \ln(2x) + \ln(2x^2) + \ln(2x^3)$

$= \ln(2x \times 2x^2 \times 2x^3) = \ln(8x^6) = 6$

$8x^6 = e^6 \quad \therefore x = \sqrt[6]{\frac{1}{8}e^6} = \frac{1}{\sqrt{2}}e$

d) $fg(x) = f(\ln(2x)) = e^{2\ln(2x)} + k^2 = (2x)^2 + k^2 = 4x^2 + k^2$

e) $4x^2 + k^2 = 2k^2 \Rightarrow 4x^2 = k^2 \Rightarrow x^2 = \frac{1}{4}k^2 \quad \therefore x = \pm\frac{1}{2}k \quad \therefore x = \frac{1}{2}k$

7.

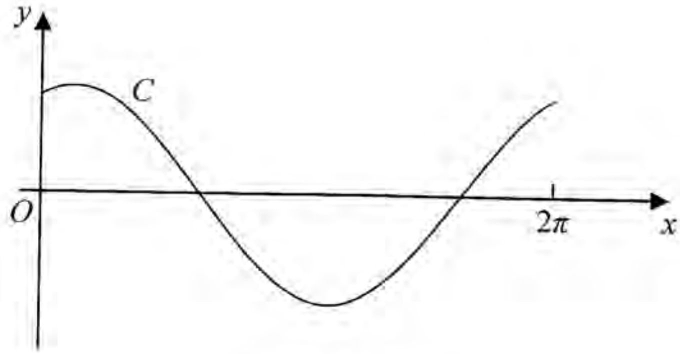


Figure 1

Figure 1 shows the curve C, with equation $y = 6\cos x + 2.5\sin x$ for $0 \leqslant x \leqslant 2\pi$

(a) Express $6\cos x + 2.5\sin x$ in the form $R\cos(x - \alpha)$, where R and α are constants with $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give your value of α to 3 decimal places.

(3)

(b) Find the coordinates of the points on the graph where the curve C crosses the coordinate axes.

(3)

A student records the number of hours of daylight each Sunday throughout the year. She starts on the last Sunday in May with a recording of 18 hours, and continues until her final recording 52 weeks later.

She models her results with the continuous function given by

$$H = 12 + 6\cos\left(\frac{2\pi t}{52}\right) + 2.5\sin\left(\frac{2\pi t}{52}\right), \quad 0 \leqslant t \leqslant 52$$

where H is the number of hours of daylight and t is the number of weeks since her first recording.

Use this function to find

(c) the maximum and minimum values of H predicted by the model,

(3)

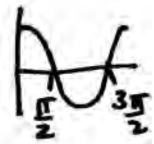
(d) the values for t when $H = 16$, giving your answers to the nearest whole number.

[*You must show your working. Answers based entirely on graphical or numerical methods are not acceptable.*]

(6)

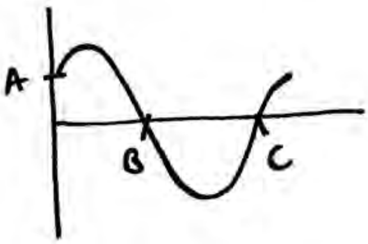
7a) $R\cos(x-\alpha) = R\cancel{\cos x}\cos\alpha + R\sin x\sin\alpha$

$6\cancel{\cos x} \quad + 2.5\sin x$



$\dfrac{R\sin\alpha = 2.5}{R\cos\alpha = 6} \Rightarrow \tan\alpha = \frac{5}{12} \quad \alpha = 0.395$

$R^2 = 2.5^2 + 6^2 \quad \therefore R = 6.5 \qquad 6.5\cos(x - 0.395\ldots)$



$x=0 \quad y=6 \quad A(0,6) \qquad B, C \xrightarrow{} 0.395\ldots$

$B(1.97, 0)$

$C(5.11, 0)$

c) $H = 12 + 6.5\cos(x - 0.395\ldots)$ when $x = \frac{2\pi t}{52}$

$6.5\cos(x - 0.395)$ ↗ max $= 6.5$ when $x = 0.395, 2\pi + 0.395\ldots$

↘ min $= -6.5$ when $x = \frac{\pi}{2} + 0.395, \frac{3\pi}{2} + 0.395\ldots$

$H_{max} = 18.5 \qquad H_{min} = 5.5$

d) $16 = 12 + 6.5\cos(x - 0.395) \Rightarrow \cos(x - 0.395) = \frac{4}{6.5}$

$\therefore x - 0.395 = 0.9079, 5.375 \rightarrow +2\pi$ etc.

$\therefore x = 1.3027, 5.77, \ldots$

$\frac{2\pi t}{52} =$ ↗

$(\times 52) \div (2\pi) \quad t = 10.78 \;;\; 47.75$

$t = 11, 48$